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Parastatistics and Dirac Brackets

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Abstract

Parastatistics (parafields) has been used in relation to several models of physical systems like the quark and the nuclear shell models. However, the physics of parafields is not completely clear. If classical para-Bose or para-Fermi variables could be constructed, then because of the correspondence principle some traces of the corresponding quantum properties could be found at the classical limit. In this way, by studying the simplest c-number systems some hints for the quantum of parafields could be expected.

We introduce and discuss classical paravariables. We construct c-number para-Fermi variables in terms of coupled classical oscillators. Several similarities to the corresponding quantum case are observed. The results support Cusson's remark that systems described in terms of parastatistics may really be composite systems.

1. *Introduction*

Parafields, i.e., fields obeying Green's commutation relations (Green, 1953; Greenberg & Messiah, 1965), have been used in relation to several models of physical systems: (1) *The quark model* (Greenberg, 1964; Morpurgo, 1970); (2) *The nuclear pairing force model for a single j-shell* (Cusson, 1969); (3) *Spin-½ oscillators with spin-orbit interaction* (Cusson, 1969). The statistics of a system described in terms of parafields is called parastatistics. Parafields are classified as para-Bose or para-Fermi fields. The relations

$$[[\overset{+}{a}_{i,op}, a_{j,op}]_{\pm}, \overset{+}{a}_{k,op}]_{-} = 2\delta_{kj}\overset{+}{a}_{i,op} \tag{1.1a}$$

$$[[a_{i,op}, a_{j,op}]_{\pm}, a_{k,op}]_{-} = 0 \tag{1.1b}$$

define the Green algebra of para-Bose variables† for the upper sign and para-Fermi variables for the lower. Bose (Fermi) variables are particular cases of para-Bose and para-Fermi variables, respectively.

† We use the word *variable* to encompass both the variables a_i with discrete i and the field variables.

In spite of the above examples and the study of its general properties, the physics of parastatistics in quantum field theory is by no means transparent. On the other hand, because of the correspondence principle, some traces of the physics of parafields can be suspected in the classical limit. In such a case they could be used as hints for the quantum case. That is why we consider it worthwhile to study the classical analogues of paravariables, and the purpose of the present paper is to make the first step in this direction.

2. *Resumé of Brackets and Quantisation*

The classical brackets

$$\{g, h\}_{\mp} = \frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_k} \mp \frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_k} \tag{2.1}$$

or $\mp$*Poisson brackets* are to be used in the quantisation rule when the phase space variables q_i and $p_i = \partial L/\partial \dot{q}_i$ are *independent*. The +Poisson bracket was introduced by Droz-Vincent (1966) under a more general expression. In terms of these brackets classical mechanics can be formulated (Droz-Vincent, 1966; Franke & Kálnay, 1970) so that the balance between antisymmetric (−) and symmetric (+) formalism is the same for the classical as for the quantum case (bosons and fermions).

Let us consider now those systems (such as the *gravitational field* and the *field of the relativistic electron*) whose phase space variables are not all independent (*phase space constraints*). There exists a subset of these systems such that for the quantisation rule the $\mp$*Dirac brackets*

$$\{g, h\}^*_{\mp} = \{g, h\}_{\mp} - \sum_{a,b} \{g, \theta^a\}_{\mp}\, c^{\mp}_{ab}\{\theta^b, h\}_{\mp} \tag{2.2}$$

(Dirac, 1950, 1964; Franke & Kálnay, 1970) should be used instead of the $\mp$Poisson brackets. Otherwise contradictions arise. In equation (2.2) the set of the $\theta^a(q,p)$, which are constraints,

$$\theta^a \approx 0 \tag{2.3}$$

is a maximal subset of all the phase space constraints.‡ The maximality refers to the requirement that the matrix $\{\theta^a, \theta^b\}_{\mp}$ be non-singular. The inverse matrix equals $c^{\mp}_{ab}$. Such θ^a-s are called irreducible $\mp$second-class constraints. The sets of the (−) and the (+) second-class constraints may be different. When no information on the existence (or not) of $\mp$second-class constraints is given, we denote the $\mp$Poisson bracket or the $\mp$Dirac bracket by

$$\{\ ,\ \}^B_{\mp}$$

according to which should be used.

One of the reasons for introducing the symmetric brackets $\{\ \}^B_+$ was to put on an equal footing the quantisation of Bose and Fermi systems (Droz-

‡ The weak equality $\approx$ has the sense explained by Dirac (1950, 1964).

Vincent, 1966; Franke & Kálnay, 1970; Kálnay & Ruggeri, 1972; Kálnay, to be published). Considering all the cases, the quantisation rule reads for boson (upper sign) or Fermi (lower sign) systems,

$$\xi_{\mp}\{\ ,\ \}_{\mp}^{B} \to [\ ,\ ]_{\mp}, \qquad \xi_{-} = i, \qquad \xi_{+} = \xi \tag{2.4}$$

where ξ is the parameter studied by Kálnay & Ruggeri (1972).

3. *Classical Paravariables*

3.1. *General*

Let a_i and their complex conjugates $\bar{a}_i$ be dynamical variables of a classical system. Let $\{\ ,\ \}_{\pm}^{B}$ and $\{\ ,\ \}_{\pm}^{B'}$ be $\pm$Poisson or Dirac brackets according to the absence or existence of $\pm$second-class constraints. Taking into account the quantisation rule (2.4) it follows that

$$\{\{\bar{a}_i, a_j\}_{\pm}^{B}, \bar{a}_k\}_{-}^{B'} = 2K_{\pm}\,\delta_{kj}\,\bar{a}_i \tag{3.1.1a}$$

$$\{\{\bar{a}_i, a_j\}_{\pm}^{B}, a_k\}_{-}^{B'} = -2K_{\pm}\,\delta_{ik}\,a_j \tag{3.1.1b}$$

$$\{\{a_i, a_j\}_{\pm}^{B}, a_k\}_{-}^{B'} = 0 \tag{3.1.1c}$$

$$\{\{\bar{a}_i, \bar{a}_j\}_{\pm}^{B}, a_k\}_{-}^{B'} = 0 \tag{3.1.1d}$$

$$K_{\pm} = -i\xi_{\pm}^{-1}, \qquad \text{i.e. } K_{+} = -i\xi^{-1},\ K_{-} = -1 \tag{3.1.1e}$$

as a possible classical partner of equations (1.1) and their hermitian conjugates.† We call, respectively, a_i, $\bar{a}_i$ para-Bose (para-Fermi) variables in the case of upper (+) and lower (−) signs. Of course, the terminology is formal, since it only regards the algebraic structure of equations (1.1) and (3.1.1) and has nothing to do with occupation numbers.

Note that in the para-Fermi case $B' = B$ is the only possibility.

3.2. *Remarks*

(A) In the classical limit a system may be simultaneously Bose-like and Fermi-like‡ (see Droz-Vincent, 1966; Franke & Kálnay, 1970; Kálnay, to be published). However, *if the c-number system is described in a quantum-like language, different quantum-like languages (such as Bose-like and Fermi-like) should be carefully distinguished in order to avoid inconsistencies.* Example: In classical mechanics a B^* algebra§ can be introduced with a

† Equations (3.1.1a) and (3.1.1c) correspond to equations (1.1). Equations (3.1.1b) and (3.1.1d) correspond to the hermitian conjugates of equations (1.1) and can be deduced from equations (3.1.1a) and (3.1.1c) if $\overline{\{g,h\}_{\pm}^{B}} = \{\bar{g},\bar{h}\}_{\pm}^{B}$; moreover, ξ *must then be real.* However, when working with complex variables, important cases appear [as, for example, the Dirac field (Kálnay, to be published)] where the distributivity of complex conjugation on Dirac brackets is violated without introducing inconsistencies in the formalism.

‡ This is not strange: different sequences in, e.g., the real plane can have as their limit one and the same point.

§ For the terminology see, e.g., Rickarts (1960). A B^* algebra is an abstract C^* algebra which, in turn, is the algebra of quantum mechanics.

*product (cf. Alonso, Kálnay & Mujica, 1970; Kálnay, Alonso, Franke & Mujica, to be submitted for publication) such that

$$i\delta_{ij} = i\{a_i, \tilde{a}_j\}_-^B = a_i * \tilde{a}_j - \tilde{a}_j * a_i \tag{3.2.1a}$$

$$0 = i\{a_i, a_j\}_-^B = a_i * a_j - a_j * a_i, \qquad \text{etc.} \tag{3.2.1b}$$

The algebra is associative. Similarly, another B^* algebra with an associative product $*'$ can be defined such that

$$\xi\delta_{ij} = \xi\{a_i, \tilde{a}_j\}_+^{B'} = a_i *' \tilde{a}_j + \tilde{a}_j *' a_i \tag{3.2.2a}$$

$$0 = \xi\{a_i, a_j\}_+^{B'} = a_i *' a_j + a_j *' a_i, \qquad \text{etc.} \tag{3.2.2b}$$

For both algebras the involution is the complex conjugation. The classical system can be described in terms of any of these algebras: they offer two languages for one physical system. However, if both languages are confused by considering them as identical,

$$* = *',$$

then it can be shown that contradiction arises.

(B) For similar reasons, different parastatistics-like descriptions of a c-number system (such as para-Bose-like and para-Fermi-like) cannot be mixed unless specifically proved that the mixing can be done consistently. If this happens, probably it is because the same happens at the quantum level (cf. Section 4.3).

(C) Again for similar reasons, if, for example, a Bose-like description [equations (3.2.1)] is used for the classical system, then the expression

$$G =^{df} a_i * \tilde{a}_j + \tilde{a}_j * a_i$$

with the *product induced through equations (3.2.1) should generally be used instead of

$$G = \xi_+\{a_i, \tilde{a}_j\}_+^{B'}$$

for the classical analogue of the quantum relation

$$G_{op} = a_{i,op}\overset{+}{a}_{j,op} + \overset{+}{a}_{j,op} a_{i,op}$$

Exceptions may arise, but their legitimacy should be proved in each case.

(D) Similar care must be taken for parastatistics-like descriptions of a c-number system. If trilinear bracket expressions [like those of equations (3.1.1)] are used for a c-number system as classical analogues of quantum trilinear commutator expressions [like those of equations (1.1)], then the bilinear bracket expression

$$k_\pm =^{df} \xi_\pm\{g, h\}_\pm^{B''}$$

cannot be used as the classical analogue of

$$k_{\pm,op} =^{df} [g_{op}, h_{op}]_\pm$$

unless specifically proved that this can be done.

(E) *It is easy to get contradictions in the classical limit if remarks A–D are not taken into account, specially for para-Bose or para-Fermi classical variables.*

(F) The irreducible representations of a quantum para-Bose or para-Fermi algebra with unique vacuum state $|0\rangle$ are known to be labelled by a natural number $p = 1, 2, 3, \ldots$ (the order of parastatistics) which is such that

$$a_{i,op} \overset{+}{a}_{j,op} |0\rangle = p\delta_{ij} |0\rangle \tag{3.2.3}$$

This involves not only the operators $a_{i,op}$, $\overset{+}{a}_{j,op}$ but also the Fock state vector space. Consequently, the order of parastatistics seems not to be easily translated to the classical limit. However, a first step will be shown in Section 4.3.

4. *An Example of c-Number para-Fermi Variables*

4.1. *Results of Previous Work*

In a paper by Kálnay & Ruggeri (1972) the Lagrangian†

$$L_{\xi'}(b, \dot{b}) = \sum_{A,B,r} [\dot{b}_{Ar}\, \sigma'_{AB}(\xi')\, b_{Br} - (1/2)\, \omega_r\, b_{Ar}\, \sigma_{AB}\, b_{Br}] - \mathscr{U}(b) \tag{4.1.1}$$

was studied. Here

$$A, B = \mathrm{I}, \mathrm{II}, \qquad r, s = 1, 2, \ldots, N$$

$$b_{\mathrm{I}r} = b_r, \qquad b_{\mathrm{II}r} = \bar{b}_r \tag{4.1.2}$$

are the configuration (complex) co-ordinates and the only non-zero matrix elements of σ and $\sigma'(\xi')$ are

$$\sigma_{\mathrm{I\,II}} = \sigma_{\mathrm{II\,I}} = 1, \qquad \sigma'_{\mathrm{I\,II}}(\xi') = (\xi' - i)/2, \qquad \sigma'_{\mathrm{II\,I}}(\xi') = (\xi' + i)/2 \tag{4.1.3}$$

The ω_r are real constants and ξ' is an arbitrary non-zero complex parameter. It was shown that the physical system described by the Lagrangian (4.1.1) has the following properties:

(i) Phase space constraints exist, Dirac's generalisation of Hamiltonian mechanics applies and Dirac brackets must be used for the quantisation.

(ii) The system is a set of coupled oscillators of frequencies ω_r:‡

$$\dot{b}_r = i\omega_r b_r + i\partial\mathscr{U}/\partial\bar{b}_r \tag{4.1.4}$$

(iii) The +Dirac bracket of dynamical variables was computed. For $g(b, \bar{b})\ h(b, \bar{b})$ which are *only* functions of the configuration variables, equation (3.13) of Kálnay & Ruggeri (1972) implies

$$\{g, h\}_+^* = \xi'^{-1} \sum_{MM'm} (\partial g/\partial b_{Mm})\, \sigma_{MM'} (\partial h/\partial b_{M'm}) \tag{4.1.5}$$

† Because of the Bose language we shall use later on, here we denote by b_{Ar} the variables which by Kálnay & Ruggeri (1972) were called a_{Ar}.

‡ The sum convention is nowhere used.

In particular, the equalities

$$\xi'\{b_r, \bar{b}_s\}_+^* = \delta_{rs} \tag{4.1.6a}$$

$$\xi'\{b_r, b_s\}_+^* = \xi'\{\bar{b}_r, \bar{b}_s\}_+^* = 0 \tag{4.1.6b}$$

show the system as a formal classical limit of a quantum Fermi system. [Cf. equation (2.4), with $\xi' = \xi$.]

(iv) It was mentioned that the same system is a formal classical limit of a quantum Bose system.

4.2. *The para-Fermi Example*

The construction starts from remark (iv) so that first of all we put it into an explicit form. In the same way as was done by Kálnay & Ruggeri (1972) for the plus Dirac brackets, we now compute the minus Dirac brackets of functions $g(b,\bar{b})$, $h(b,\bar{b})$ of *configuration* space variables. The result is

$$\{g, h\}_-^* = -i \sum_{MM'm} (\partial g/\partial b_{Mm})\, \epsilon_M\, \sigma_{MM'} (\partial h/\partial b_{M'm})$$

$$\epsilon_{\mathrm{I}} = +1, \qquad \epsilon_{\mathrm{II}} = -1 \tag{4.2.1}$$

In particular,

$$\{b_{Rr}, b_{Ss}\}_-^* = -i\epsilon_R\, \sigma_{RS}\, \delta_{rs} \tag{4.2.2}$$

i.e.,

$$i\{b_r, \bar{b}_s\}_-^* = \delta_{rs} \tag{4.2.3a}$$

$$i\{b_r, b_s\}_-^* = i\{\bar{b}_r, \bar{b}_s\}_-^* = 0 \tag{4.2.3b}$$

By comparison with equations (2.4) we see that equations (4.2.3) are the classical partners of the quantum Bose commutation relations.

We stress remark D of Section 3.2 in order to avoid misunderstandings and inconsistencies. The system governed by the Lagrangian (4.1.1) is the classical limit of Bose *and* Fermi systems but, whenever quantum analogues are looked for at the classical level, one cannot use Bose-like *and* Fermi-like descriptions simultaneously. Kálnay & Ruggeri (1972) described the classical system in terms of the classical limit of a Fermi language. *In the present paper the same system will be described in terms of the classical limit of a Bose language* and we should not mix both languages.

The reason why we need a Bose-like description of the classical system (4.1.1), while our purpose is to look for para-Fermi variables, is the following. In the quantum cases it was shown (Kademova, 1970a; Kademova & Kálnay, 1970; Kálnay, Mac Cotrina & Kademova, to be published; Kálnay & Kademova, to be submitted for publication) that para-Fermi operators can be realised in terms of polynomials of Bose operators. We ask if such a result can be translated to the c-number level. If we are lucky, then this would solve the problem of the construction of classical para-Fermi variables and its properties could be studied in the easiest classical limit.

Lemma: Let α^i, $i = 1, 2, 3, \ldots$, be $N \times N$ matrices and define

$$A^{\alpha_i}(b, \bar{b}) = \sum_{r,s=1}^{N} \alpha^i_{rs} \bar{b}_r b_s \tag{4.2.4}$$

Then

$$\{A^{\alpha_i}, A^{\alpha_j}\}^*_\pm = \xi'^{-1}_\pm \sum_{r,s=1}^{N} ([\alpha^i, \alpha^j]_\pm)_{rs} \bar{b}_r b_s, \qquad \xi'_- = i,\ \xi'_+ = \xi' \tag{4.2.5}$$

Proof: Use equations (4.1.5) and (4.2.1). □

The Lemma resembles the Proposition 2 of Kademova (1970b) which stands for the quantum case.

In what follows we choose $N = 2^M$, $M = 1, 2, 3, \ldots$

Let F_i, $i = 1, 2, \ldots, M$ be $N \times N$ matrices and $\overset{+}{F}_i$ their hermitian conjugates which span a Fermi algebra

$$[F_i, \overset{+}{F}_j]_+ = \delta_{ij} I \tag{4.2.6a}$$

$$[F_i, F_j]_+ = [\overset{+}{F}_i, \overset{+}{F}_j]_+ = 0 \tag{4.2.6b}$$

For definiteness, we can use the Fermi matrices given by Kademova & Kálnay (1970) for the finite case and the matrices introduced by Kálnay, Mac Cotrina & Kademova (to be published) for the infinite case. We define

$$f_i(b, \bar{b}) = A^{F_i}(b, \bar{b}), \qquad i = 1, 2, \ldots, M \tag{4.2.7}$$

i.e.,

$$f_i = \sum_{r,s=1}^{N} (F_i)_{rs} \bar{b}_r b_s \tag{4.2.8a}$$

$$\bar{f}_i = \sum_{r,s=1}^{N} (\overset{+}{F}_i)_{rs} \bar{b}_r b_s, \qquad i = 1, 2, \ldots, M \tag{4.2.8b}$$

These resemble the quantum formulae.† (Cf. Kademova, 1970a; Kademova & Kálnay, 1970; Kálnay, Mac Cotrina & Kademova, to be published; Kálnay & Kademova, to be submitted for publication.)

Theorem: The Bose polynomials $f_i, \bar{f}_i$ are c-number para-Fermi variables.

Proof: Because of the Lemma, we have

$$\{\{\bar{f}_i, f_j\}^*_-, \bar{f}_k\}^*_- = -\sum_{rs} ([[\overset{+}{F}_i, F_j]_-, \overset{+}{F}_k]_-)_{rs} \bar{b}_r b_s \tag{4.2.9a}$$

$$\{\{f_i, f_j\}^*_-, f_k\}^*_- = -\sum_{rs} ([[F_i, F_j]_-, F_k]_-)_{rs} \bar{b}_r b_s \tag{4.2.9b}$$

But Fermi matrices are particular cases of para-Fermi matrices (Green, 1953), so that equations (1.1) (with the lower sign) hold for them, i.e.,

$$[[\overset{+}{F}_i, F_j]_-, \overset{+}{F}_k]_- = 2\delta_{kj} \overset{+}{F}_i \tag{4.2.10a}$$

† When comparing with the quantum case it should be remembered that $TrF = 0$ so that non-commutative products of Bose operators do not appear on the quantum analogue of equation (4.2.8) (C.f, Lemma A in Kálnay & Mac Cotrina, submitted for publication).

$$[[F_i, F_j]_-, F_k]_- = 0 \tag{4.2.10b}$$

so that

$$\{\{\bar{f}_i, f_j\}^*_-, \bar{f}_k\}^*_- = -2\delta_{kj}\bar{f}_i \tag{4.2.11a}$$

$$\{\{f_i, f_j\}^*_-, f_k\}^*_- = 0 \tag{4.2.11b}$$

These are identical to equations (3.1.1a) and (3.1.1c) with the lower sign and $\{\ ,\ \}^B_- = \{\ ,\ \}^{B'}_- = \{\ ,\ \}^*_-$ [property (i) of Section 4.1]. Equations (3.1.1b) and (3.1.1d) are deduced in the same way but by using, instead of equations (1.1), their hermitian conjugates. □

Note: The variables $f_i, \bar{f}_i$ are *not* para-Bose variables.

Proof: Because of the Lemma the equation

$$\{\{\bar{f}_i, f_j\}^*_+, \bar{f}_k\}_- = -i\xi'^{-1} \sum_{r,s} ([[\overset{+}{F_i}, F_j]_+, \overset{+}{F_k}]_-)_{rs} \bar{b}_r b_s \tag{4.2.12}$$

is obtained. Then by using equation (4.2.6a) it results in

$$\{\{\bar{f}_i, f_j\}^*_+, \bar{f}_k\}_- = 0 \tag{4.2.13}$$

which contradicts equation (3.1.1a) with the upper sign. □

4.3. *On the Classical Order of Parastatistics*

In remark F of Section 3.2 the difficulties of extending to the classical level the order p of parastatistics were briefly discussed. However, for $p = 1$ the situation is simpler because $p = 1$ para-Fermi statistics is Fermi statistics (Green, 1953). This means that the classical limit of $p = 1$ para-Fermi statistics should be a system in which equations (4.2.11) should co-exist with

$$\xi\{f_i, \bar{f}_j\}^*_+ = \delta_{ij} \tag{4.3.1a}$$

$$\xi\{f_i, f_j\}^*_+ = \xi\{\bar{f}_i, \bar{f}_j\}^*_+ = 0 \tag{4.3.1b}$$

[Cf. equation (2.4) and property (i) of Section 4.1.]

Theorem: The c-number para-Fermi variables $f_i, \bar{f}_i$ given by equations (4.2.8) are also c-number Fermi variables when the underlying Bose system is in such a state that

$$\sum_r \bar{b}_r b_r = 1 \tag{4.3.2}$$

and $\xi' = \xi$ is used.

Proof: Use equations (4.2.6), (4.2.8), (2.4) and the Lemma. □

Notes:

(1) It is quite surprising that in the quantum case any irreducible representation of the para-Fermi algebra which is also an irreducible representation of a Fermi algebra can be realised in the one-particle Bose states (Kademova, 1970a; Kademova & Kálnay, 1970; Kálnay, Mac

Cotrina & Kademova, to be published; Kálnay & Kademova, to be submitted for publication), i.e. in those quantum Bose states $|\rangle$ such that

$$\sum_r \overset{+}{b}_{r,op} b_{r,op} |\rangle = |\rangle \tag{4.3.3}$$

which strongly resembles the classical formula (4.3.2).

(2) However, the resemblance is not complete because of the c-number terms which could be added to the right-hand side of equation (4.3.3) by simultaneously changing the order of the operators on the left-hand side.

(3) For an arbitrary state we have (putting $\xi' = \xi$),

$$\xi\{f_i, \bar{f}_j\}_+^* = \delta_{ij} \sum_r \bar{b}_r b_r, \qquad \xi\{f_i, f_j\}_+^* = 0 \tag{4.3.4}$$

That is why it could be thought that even if equation (4.3.2) is violated, new variables

$$\psi_i(b, \bar{b}) = \left(\sum_r \bar{b}_r b_r\right)^{-1/2} f_i(b, \bar{b}) \tag{4.3.5}$$

could always be introduced such that ψ_i, $\bar{\psi}_i$ be Fermi variables. However, it is easily shown that to require (with $\xi' = \xi$)

$$\xi\{\psi_i, \bar{\psi}_i\}_+^* = 1$$

leads to contradiction unless in physically uninteresting cases all para-Fermi variables equal zero.

5. *Elementarity or Higher Order of para-Fermi Statistics?*

After considering several quantum parastatistics systems Cusson (1969) suggested that '... paraquanta may represent "composite" particles ...'. Kademova (1970a), Kademova & Kálnay (1970), Kálnay, Mac Cotrina & Kademova (to be published), Kálnay & Kademova (to be submitted for publication) have shown that for the quantum case the complete Fock space of any irreducible representation of para-Fermi statistics of order p of parastatistics can be realised in the Bose–Fock subspace of p bosons. With the exception of fermions, all other parafermions require several bosons for the above realisation. This seems to confirm Cusson's hypothesis.

By comparing the above conclusions with the results of Section 4 we see that, as suspected in the Introduction, traces (really stronger than could be expected) of the physics of paravariables can be found in the classical limit. The possible importance of this fact was explained in the Introduction.

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